Trinity College

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine the values of the real constants b and c if $z = 1 + 3i$ is a solution of the equation $z^2 + bz + c = 0$. (3 marks)

Solution
$b = -\big((1 + 3i) + (1 - 3i)\big)$ $= -2$ $c = (1 + 3i)(1 - 3i)$ $= 10$
Specific behaviours
✓ indicates other (conjugate) solution ✓ states value of b ✓ states value of c

(b) Express the real quadratic polynomial $z^2 - 4z + 8$ as a product of its linear factors. (3 marks)

Solution
$z^2 - 4z + 8 = (z - 2)^2 - 4 + 8$ $= (z - 2)^2 - (-4)$ $= (z - 2)^2 - (2i)^2$ $= (z - 2 - 2i)(z - 2 + 2i)$
Specific behaviours
✓ complete square ✓ express as difference of squares ✓ write as required

Question 2 **(6 marks)**

(a) A set of real numbers is given by $\{\sqrt{2},\ 3.\overline{14},\ \pi,\ \sqrt[3]{14}\}$. Clearly show that one of the numbers in the set is rational. (3 marks)

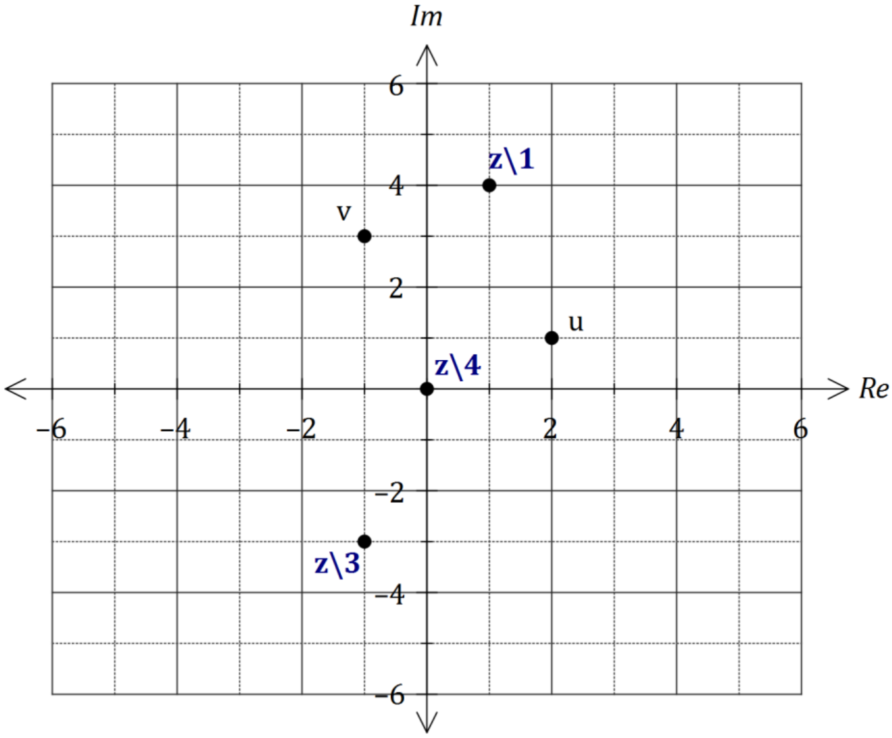
Solution
let $x = 3.\overline{14}$ Then $100x - x = 314.\overline{14} - 3.\overline{14}$ $99x = 311$ $x = \frac{311}{99}$ and hence is rational
Specific behaviours
✓ chooses rational number   ✓ indicates use of $100x - x$ ✓ writes as rational

(b) Show that if n is one more than a multiple of three, then n^2 will also be one more than a multiple of three, where $n \in \mathbb{Z}$. (3 marks)

Solution
Let $n = 3k + 1, k \in \mathbb{Z}$ Then $n^2 = 9k^2 + 6k + 1$ $= 3(3k^2 + 2k) + 1$ Hence true
Specific behaviours
✓ writes n in required form   ✓ squares n ✓ writes n^2 in required form

Question 3 **(4 marks)**

The complex numbers u and v are shown in the complex plane below.



Plot and label the following complex numbers:

(a) $z_1 = u + v$. (1 mark)

(b) $z_3 = \bar{v}$. (1 mark)

(c) $z_4 = \overline{u + v} - \bar{u} - \bar{v}$. (2 mark)

Solution
See graph
Specific behaviours
✓ z_1 ✓ z_3 ✓ z_4

Question 4 **(8 marks)**

Relative to the origin O, the points A, B and C have position vectors $\mathbf{a} = 5\mathbf{i} - 6\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = -8\mathbf{i} + 15\mathbf{j}$ respectively.

(a) Determine in Cartesian form

(i) the vector $\overrightarrow{AB}$. (1 mark)

Solution
$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ $= -4\mathbf{i} + 3\mathbf{j}$
Specific behaviours
✓ subtracts position vectors

(ii) a vector $\mathbf{d}$, parallel to $\overrightarrow{AB}$ and of magnitude $\sqrt{5}$. (3 marks)

Solution
$\lvert\overrightarrow{AB}\rvert = 5$ $\mathbf{d} = \frac{\sqrt{5}}{5}(-4\mathbf{i} + 3\mathbf{j})$
Specific behaviours
✓ states magnitude ✓ indicates unit vector ✓ states required vector (in either direction)

(b) If $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, determine the values of the constants λ and μ. (4 marks)

Solution
$\mathbf{i}$-coeff: $5\lambda + \mu = -8$ $\mathbf{j}$-coeff: $-6\lambda - 3\mu = 15$ $15\lambda + 3\mu = -24$ $-6\lambda - 3\mu = 15$ $9\lambda = -9 \Rightarrow \lambda = -1$ $\mu = -8 + 5 = -3$
Specific behaviours
✓ uses coefficients to form equations ✓ uses elimination or substitution ✓ states λ ✓ states μ

Question 5 **(9 marks)**

Let $A = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -3 & -1 \end{bmatrix}$.

(a) Determine

(i) $3A - B$. (2 marks)

Solution
$3A - B = \begin{bmatrix} 24 & 9 \\ 15 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -3 & -1 \end{bmatrix}$ $= \begin{bmatrix} 22 & 5 \\ 18 & 7 \end{bmatrix}$
Specific behaviours
✓ multiple of A ✓ difference

(ii) BA. (2 marks)

Solution
$BA = \begin{bmatrix} 2 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ $= \begin{bmatrix} 36 & 14 \\ -29 & -11 \end{bmatrix}$
Specific behaviours
✓ at least two elements correct   ✓ all elements correct

(iii) A^{-1}. (2 marks)

Solution
$\lvert A \rvert = 16 - 15 = 1$ $A^{-1} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$
Specific behaviours
✓ indicates use of determinant   ✓ correct inverse

(b) Use a matrix method to solve the system of equations $8x + 3y = 10$ and $5x + 2y = 7$. (3 marks)

Solution
$\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix} \begin{bmatrix} 10 \\ 7 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$
Specific behaviours
✓ writes as matrix equation   ✓ shows correct use of inverse   ✓ states solution

Question 6 **(8 marks)**

(a) Determine the acute angle θ in each of the following cases:

(i) $\cos\theta = \sin 38°$. (2 marks)

Solution
$\theta = 90 - 38$
$\theta = 52°$
Specific behaviours
✓ uses complement
✓ states angle

(ii) $\sec\theta = \operatorname{cosec} 100°$. (2 marks)

Solution
$\dfrac{1}{\cos\theta} = \dfrac{1}{\sin 100} = \dfrac{1}{\sin 80}$
$\theta = 90 - 80 = 10°$
Specific behaviours
✓ sin in first quadrant
✓ states angle

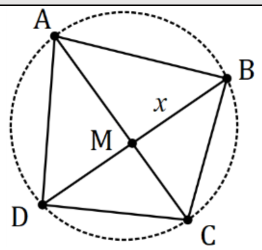
(b) Prove that $\tan x + \sec x = \dfrac{\cos x}{1 - \sin x}$. (4 marks)

Solution
$\text{LHS} = \dfrac{\sin x}{\cos x} + \dfrac{1}{\cos x}$
$= \dfrac{(\sin x + 1)}{\cos x} \times \dfrac{\cos x}{\cos x}$
$= \dfrac{\cos x\,(1 + \sin x)}{1 - \sin^2 x}$
$= \dfrac{\cos x\,(1 + \sin x)}{(1 - \sin x)(1 + \sin x)}$
$= \dfrac{\cos x}{1 - \sin x}$
Specific behaviours
✓ writes LHS as single fraction
✓ multiplies by $\cos x \div \cos x$
✓ uses Pythagorean identity
✓ factorises denominator and simplifies

Question 7 **(5 marks)**

Cyclic quadrilateral $ABCD$ has diagonals AC and BD that intersect at M. Given that $AM = 6$ cm, $CM = 8$ cm and $BD = 16$ cm, determine the smallest possible length of BM.

Solution



Let $BM = x$, then $DM = 16 - x$

$$x(16 - x) = 6 \times 8 = 48$$

$$x^2 - 16x - 48 = 0$$
$$(x - 8)^2 = 16$$

$$x = 8 \pm 4 \Rightarrow BM = 4 \text{ cm}$$

Specific behaviours

✓ annotated diagram
✓ expressions for BM and DM
✓ uses intersecting chord theorem to form equation
✓ factors equation
✓ states required solution

Question 8 **(6 marks)**

Let z_1 and z_2 be complex numbers such that $2z_1 + 3z_2 = 7$ and $z_1 + iz_2 = 4 + 4i$.

Determine z_1 and z_2 in the form $z = a + bi$, where $a, b \in \mathbb{Z}$.

Solution
$2z_1 + 3z_2 = 7$ $2z_1 + 2iz_2 = 8 + 8i$ $z_2(3 - 2i) = (-1 - 8i)$ $z_2 = \frac{(-1 - 8i)}{(3 - 2i)}$ $z_2 = \frac{(-1 - 8i)}{(3 - 2i)} \times \frac{(3 + 2i)}{(3 + 2i)}$ $z_2 = \frac{13 - 26i}{13}$ $z_2 = 1 - 2i$ $z_1 = 4 + 4i - i(1 - 2i)$ $z_1 = 2 + 3i$
Specific behaviours
✓ eliminate z_1 ✓ express z_2 as quotient ✓ realise denominator ✓ state z_2 ✓ substitute for z_1 ✓ state z_1

End of questions

Additional working space

Question number: _________